

Aging Random Walks

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Aging refers to the property of two-time correlation functions to decay very slowly on (at least) two time scales. This phenomenon has gained recent attention due to experimental observations of the history dependent relaxation behavior in amorphous materials (“Glasses”) which pose a challenge to theorist. Aging signals the breaking of time-translational invariance and the violation of the fluctuation dissipation theorem during the relaxation process. But while the origin of aging in disordered media is profound, and the discussion is clad in the language of a well-developed theory, systems as simple as a random walk near a wall can exhibit aging. Such a simple walk serves well to illustrate the phenomenon and some of the physics behind it.

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I. INTRODUCTION

Aging is an important phenomenon observed experimentally in glassy materials where relaxation behavior depends on the previous history of the system [1]. As a model for a glass Edwards and Anderson [2] introduced an Ising system where the uniform coupling $J > 0$ between neighboring spins is replaced by random numbers $J_{i,j}$, drawn from some distribution, which are placed on each bond. The properties of such a disordered system are quite different

when the couplings are drawn from, say, a gaussian distribution with zero mean, compared to those of the ferro-magnetic Ising model with uniform couplings $J > 0$: While (for $d \geq 2$) the uniform Ising model reaches an ordered phase below a critical temperature T_c with a nonzero spontaneous magnetization as order parameter, no such order emerges in the random bond model. Yet, below a certain temperature T_g , the “glass transition,” more and more spins lose their mobility and freeze into place, but without any collectively preferred direction. Thus, while no macroscopic order parameter emerges, microscopically the state of the system is ultimately highly (auto-)correlated. The relaxation process towards such a state is naturally extremely slow due to the inherent frustration created by the bond distribution, and each spin has to “negotiate” its orientation with ever distant neighbors to further lower the collective energy within their domain.

Consider such a spin glass quenched below the glass transition T_g at time (“age”) $t_a = 0$ in the presence of a magnetic field. Throughout the sample, domains of various pure states develop that grow with characteristic size $L(t_a)$ [3]. After waiting a time $t_a = t_w$, the magnetic field is turned off and the system’s response in the form of its remnant magnetization is measured. Initially, the response function is only sensitive to the pure, quasi-equilibrium states in their distinct domains. But after an additional time scale related to the waiting time, t_w , the response spans entire domains and slows down when it experiences the off-equilibrium state the sample possesses as a whole. In either regime, though, the remnant magnetization decays very slowly, similar to the sequence of graphs (see Ref. [1]) in Fig. 2 below. It has been suggested [4] that a simple scaling form might hold for glasses; e.g. that the autocorrelation function for the remnant magnetization at the age $t_a = t + t_w$ of the experiment is given by

$$C(t + t_w, t_w) = t^{-\mu} f(t/t_w) \quad , \quad (1.1)$$

where the scaling function f is constant for small argument, and falls either like a power law (indication of a single aging time scale) or like a stretched exponential (indication of multiple aging time scales [5]) for large argument. In any case, time-translational invariance

in this two-time correlation function C is broken due to the fact that the relaxation process becomes dependent on its previous history in form of the waiting time t_w .

Similar quantities have been observed in a variety of phenomenological [6] and theoretical [7–10] model systems as well, and a debate is raging on the ingredients that are required for a model to capture the salient features of glassy materials [11]. While (as we will see) it is easy to produce aging behavior, the more intricate results from temperature cycling experiments [12] are much harder to reproduce, and appear to be a much more stringent indicator of the complicated phase-space structure (dubbed “rugged landscapes” [13]) real glasses possess.

In fact, a simple random walk near a (reflecting or absorbing) wall is well suited to describe the domain-growth picture used above to describe the observed aging behavior. Consider a walker starting near the wall $n = 0$ at $t_a = 0$. After a waiting time $t_a = t_w$ she has explored a domain of size $L(t_w) \sim t_w^{1/2}$ off the wall. The walk now represents the correlation function for a spin at site n in the above spin glass model, and the wall is the edge of the (albeit one-sided) domain. In place of magnetization, we measure the probability $P(n, t + t_w | n, t_w)$ for the walker to return to the same site n she found herself on at time t_w . Without the wall, P is of course invariant to shifts in space and time, i. e. t_w is irrelevant and no aging behavior can be expected. In the presense of the wall, the walker will venture from the site n for small times t after t_w and find herself unconstrained (the “quasi-equilibrium” state), but when $t \sim t_w$ she is likely to encounter the wall and carry a memory of that constraint back to the site n for $t \gg t_w$. In the next section, we will solve this model for a suitably defined two-time correlation function which indeed shows the expected aging behavior.

Of course, in this model it is the wall that explicitly breaks the symmetries of the system instead of the dynamics of the process, and the observed aging behavior appears trivial. On the other hand, the dynamics of the stochastic annihilation process $A + A \rightarrow 0$, which is similar to a walk at a wall, yields identical results to those reported below [14], and the distinction between explicit and dynamical symmetry breaking is not so clear anymore.

Finally, the case of an absorbing wall is particularly revealing and illustrates the the effect of a process that is slowly dying out (i. e. its norm decays), while true aging should only be associated with intrinsic properties of a process that will sustain itself. This point has lead to some confusion recently.

II. THE RANDOM WALK MODEL

In this section, we will calculate the conditional probability $P(n, t|n_0, t_0)$ for a walker to reach a site n at time t , given that she was at site n_0 at some previous time $t_0 < t$ in the presense of either an absorbing or a reflecting wall. Then, we will compute a simple two-time correlation function (see e. g. Ref. [7] for a similar definition)

$$C(t + t_w, t_w) = \sum_n P(n, t + t_w|n, t_w)P(n, t_w|0, 0) \quad (2.1)$$

for a walker to return to a site at time $t_a = t + t_w$, given that she was at the same site at time t_w after the start of the walk at time $t_a = 0$ near the wall $n = 0$. For both boundary conditions, we find that the walk ages, i. e. shows a scaling behavior according to Eq. (1.1). Note that the breaking of spatial invariance in P due to the wall is crucial for the breaking of time-translational invariance in C : For an unconstraint walk P would be invariant in space and time, and we would find $P(n, t + t_w|n, t_w) = P(0, t|0, 0)$, and with $\sum_n P(n, t_w|0, 0) \equiv 1$, it is $C(t + t_w, t_w) = C(t, 0) = P(0, t|0, 0)$, independent of t_w . In the presense of the wall, spatial invariance is broken while time invariance for P still holds, and C merely simplifies to $C(t + t_w, t_w) = \sum_n P(n, t|n, 0)P(n, t_w|0, 0)$ which remains a function of t_w .

To simplify the algebra, we consider instead of the walk equation for P the potential problem

$$\begin{aligned} \partial_t \phi(r, t) &= \partial_r^2 \phi(r, t), \quad (r > 0, t > 0), \\ \phi(r, 0) &= \delta(r - r_0), \\ \phi(0, t) &= 0 \quad \text{or} \quad \partial_r \phi(0, t) = 0, \end{aligned} \quad (2.2)$$

where the two boundary conditions in the last line correspond to an absorbing and a reflecting wall, respectively [15]. We solve for ϕ , and identify $P(n, t|r_0, 0) = \phi_{r_0}(r, t)$ and

$$C(t + t_w, t_w) = \int_0^\infty dr \phi_r(r, t) \phi_0(r, t_w). \quad (2.3)$$

The Eq. (2.2) is easy to solve by converting to an ordinary differential equation (ODE) in r using a Laplace transform in t : $\tilde{\phi}(r, s) = \int_0^\infty dt e^{-st} \phi(r, t)$. The ODE has simple exponential solutions in two regions, $0 < r < r_0$ and $r > r_0$, whose four unknown constants are determined by the two boundary conditions at $r = 0$ and $r = \infty$, and by the two matching conditions at $r = r_0$ where ϕ is merely continuous. We find

$$\tilde{\phi}_{r_0}(r, s) = \frac{1}{\sqrt{s}} \left[e^{-\sqrt{s}|r-r_0|} \pm e^{-\sqrt{s}(r+r_0)} \right], \quad (2.4)$$

which is easily inverted using standard tables for Laplace transforms [16]

$$\phi_{r_0}(r, t) = \frac{1}{\sqrt{\pi t}} \left[e^{-\frac{(r-r_0)^2}{4t}} \pm e^{-\frac{(r+r_0)^2}{4t}} \right]. \quad (2.5)$$

In each case, the upper sign refers to reflecting boundary conditions, and the lower sign refers to the absorbing case.

A. Reflecting Boundary Conditions

Here we insert the appropriate forms of ϕ in Eq. (2.5), using the upper sign case, into Eq. (2.3) and choose $r_0 = 0$ for convenience [17]:

$$\begin{aligned} C(t + t_w, t_w) &= \frac{2}{\pi\sqrt{t}t_w} \int_0^\infty dr \left(1 + e^{-\frac{r^2}{t}} \right) e^{-\frac{r^2}{4t_w}} \\ &= \frac{2}{\sqrt{\pi}} t^{-\frac{1}{2}} f\left(\frac{t}{t_w}\right) \quad \text{with} \quad f(x) = 1 + \sqrt{\frac{x}{4+x}}. \end{aligned} \quad (2.6)$$

Thus, while the two-time correlation function does show the aging behavior according to Eq. (1.1), its scaling function is particularly trivial with $f(x \ll 1) \sim 1$ and $f(x \gg 1) \sim 2$, see Fig. 1.

B. Absorbing Boundary Conditions

Again, we insert the appropriate forms of ϕ in Eq. (2.5), using the lower sign case, into Eq. (2.3). But with absorbing boundary conditions, putting the starting position at $r_0 = 0$ would be instantly fatal for the walker. (In simulations we use $n = 1$ as starting point.) Instead, we can choose r_0 arbitrarily small and expand to leading order. Since the starting point r_0 is irrelevant for the asymptotic behavior (at large times) considered here [17], we can be sure that higher-order corrections in r_0 will have to be subdominant:

$$\begin{aligned} C(t + t_w, t_w) &= \frac{1}{\pi\sqrt{t}t_w} \int_0^\infty dr \left[1 - e^{-\frac{r^2}{t}} \right] \left[e^{-\frac{(r-r_0)^2}{4t_w}} - e^{-\frac{(r+r_0)^2}{4t_w}} \right] \\ &\approx \frac{r_0}{\pi\sqrt{t}t_w^3} \int_0^\infty dr \left[1 - e^{-\frac{r^2}{t}} \right] r e^{-\frac{r^2}{4t_w}} \\ &= \frac{r_0}{\sqrt{t_w}} \frac{2}{\pi} t^{-\frac{1}{2}} f\left(\frac{t}{t_w}\right) \quad \text{with} \quad f(x) = \frac{1}{1 + \frac{x}{4}}. \end{aligned} \quad (2.7)$$

In this case, we find more interesting scaling behavior with $f(x \ll 1) \sim 1$ and $f(x \gg 1) \sim 4/x$.

But this observed aging behavior does not consider the effect that a walk actually disappears when reaching the wall which diminishes the norm of the distribution (ϕ or P). Rather, to obtain the intrinsic properties of an infinite walk near an absorbing wall, we have to properly normalize the correlation function. To that end, we consider the two-time correlation function $C(t + t_w, t_w | \theta)$ for a walk that reaches the wall (and disappears) exactly at time $t_a = \theta$, and its generic relation to the intrinsic two-time correlation function $C^{\text{intr}}(t + t_w, t_w)$:

$$C(t + t_w, t_w | \theta) = \begin{cases} 0 & (\theta < t_w + t), \\ C^{\text{intr}}(t + t_w, t_w) & (\theta \geq t_w + t). \end{cases} \quad (2.8)$$

These quantities are related to the two-time correlation function given in Eq. (2.7) (which is usually the one that is simulated by averaging over walks of any length up to some cut-off in time): Given the probability $P_t(\theta) \sim \theta^{-\tau}$, $\tau > 1$, for the walker to reach the wall for the first time at $t_a = \theta$ (at which point the walk disappears without further contributing to the statistics in the numerical simulation), we have the identity

$$\begin{aligned}
C(t + t_w, t_w) &= \int_0^\infty d\theta C(t + t_w, t_w | \theta) P_t(\theta) \\
&\sim C^{\text{intr}}(t + t_w, t_w) (t_w + t)^{1-\tau},
\end{aligned}
\tag{2.9}$$

and thus

$$\begin{aligned}
C^{\text{intr}}(t + t_w, t_w) &\sim C(t + t_w, t_w) (t_w + t)^{\tau-1} \\
&\sim t^{-\frac{1}{2}} f\left(\frac{t}{t_w}\right) \left(1 + \frac{t}{t_w}\right)^{\tau-1}.
\end{aligned}
\tag{2.10}$$

Hence, the correct scaling function for the aging behavior of the intrinsic process is given by

$$f^{\text{intr}}(x) \sim f(x)(1+x)^{\tau-1}, \tag{2.11}$$

with $f^{\text{intr}}(x \ll 1) \sim 1$ and $f^{\text{intr}}(x \gg 1) \sim x^{\tau-2}$. Of course, $\tau = 3/2$ from the familiar first-passage time of a random walk [18], and aging remains intact although the cross-over in f^{intr} is less dramatic than before for f . In Fig. 2 we plot results for C^{intr} from numerical simulations, and in Fig. 3 we plot the scaling f^{intr} for the data in Fig. 2.

As mentioned before, the stochastic annihilation process $A + A \rightarrow 0$ is closely related to the random walk model with an absorbing wall and, indeed, the intrinsic scaling behavior found here (aside from an overall factor of \sqrt{t}) coincides with the one reported in Eq. (9) of Ref. [14] (for $t \rightarrow t + t_w$ and $\xi \rightarrow t/(t + T_w)$).

While in this walk model the aging behavior remains intact even for the intrinsic properties of the process, it is important to note that in some cases the observed aging behavior can be entirely attributed to improper normalization of the correlation functions in a process in which the norm depletes. Of course, such a situation can not be considered as aging behavior. (In fact, in a recent paper [19] this effect has even been proposed as a general explanation for aging.)

III. CONCLUSIONS AND ACKNOWLEDGMENTS

We have shown that a simple, solvable model of a random walk near a wall can exhibit aging behavior which illuminates many features that lead to aging behavior in more com-

plicated (disordered) systems, using a domain-growth picture [3]. Of course, systems with many interacting degrees of freedom such as spin glasses or folding proteins exhibit a non-trivial phase space structure [12] which leads to slow relaxation and aging behavior. Their dynamics is described merely on a coarse, phenomenological level by such a simple model. But the connection between the micro-dynamics and the macroscopic phenomena is not only beyond the scope of this article, but as well itself very much under development still. Instead of “explaining” experimental or theoretical results, this random walk model is meant to illustrate some of the questions involved. (After all, it is still rare to consider situations with broken time-translational invariance, and thus violations of the fluctuation dissipation theorem, without which two-time correlation functions would be redundant.) Furthermore, we have discussed some of the pitfalls in identifying aging behavior in systems which do not conserve the norm, and how to obtain the intrinsic features of such systems.

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FIGURES

FIG. 1. Log-log plot of the two-time correlation function $C(t + t_w, t_w)$ (arbitrary scale) from numerical simulations of a walk near a reflecting wall. Each plot contains data for $2^{i-1} \leq t_w < 2^i$ for $2 \leq i \leq 14$ where i labels each graph from bottom to top. Initially, for $t < t_w$, each correlation function falls like $1/\sqrt{t}$ with a crossover at $t \sim t_w$, after which the walk notices the effect of the (domain-)wall and the function falls like $2/\sqrt{t}$ for $t \gg t_w$. The continuous line is inserted to guide the eye and show that the plot of C for each t_w indeed falls like $t^{-1/2}$ in two separate regimes which only differ by a factor of 2.

FIG. 2. Log-log plot of the normalized, intrinsic two-time correlation function $C^{\text{intr}}(t + t_w, t_w)$ from numerical simulations of a walk near an absorbing wall. (Each plot is shifted to avoid overlaps.) Each plot contains data for $8^{i-1} \leq t_w < 8^i$ for $1 \leq i \leq 5$ where i labels each graph from bottom to top. Initially, for $t < t_w$, each correlation function falls like $1/\sqrt{t}$ with a crossover at $t \sim t_w$, after which the effect of the (domain-)wall becomes noticeable and the function falls like $1/t$.

FIG. 3. Scaling plot $f(t/t_w) \sim \sqrt{t} C^{\text{intr}}(t + t_w, t_w)$ as a function of the scaling variable t/t_w for the data in Fig. 2. All data collapses reasonably well onto a single scaling graph which is constant for small argument and falls like an inverse square-root for large argument (such as the dashed line drawn for reference).





